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Defect of characters of the symmetric group

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Abstract. Following the work of B. Külshammer, J. B. Olsson and G. R. Robinson on generalized blocks of the symmetric groups, we give a definition for the ℓ-defect of characters of the symmetric group $\mathfrak{S}_n$, where $\ell > 1$ is an arbitrary integer. We prove that the ℓ-defect is given by an analogue of the hook-length formula, and use it to prove, when $n < \ell^2$, an ℓ-version of the McKay conjecture in $\mathfrak{S}_n$.

1 Introduction

B. Külshammer, J. B. Olsson and G. R. Robinson gave in [6] a definition of *generalized blocks* for a finite group. Let G be a finite group, and denote by $\mathrm{Irr}(G)$ the set of complex irreducible characters of G. Take a union $\mathscr{C}$ of conjugacy classes of G containing the identity. Suppose furthermore that $\mathscr{C}$ is *closed*, that is, if $x \in \mathscr{C}$, and if $y \in G$ generates the same subgroup of G as x, then $y \in \mathscr{C}$. For χ, $\psi \in \mathrm{Irr}(G)$, we define the $\mathscr{C}$*-contribution* $\langle \chi, \psi \rangle_{\mathscr{C}}$ of χ and ψ by

$$\langle \chi, \psi \rangle_{\mathscr{C}} := \frac{1}{|G|} \sum_{g \in \mathscr{C}} \chi(g)\psi(g^{-1}).$$

The fact that $\mathscr{C}$ is closed implies that, for any χ, $\psi \in \mathrm{Irr}(G)$, $\langle \chi, \psi \rangle_{\mathscr{C}}$ is a rational number.

We say that χ, $\psi \in \mathrm{Irr}(G)$ belong to the same $\mathscr{C}$*-block* of G if there exists a sequence of irreducible characters $\chi_1 = \chi, \chi_2, \dots, \chi_n = \psi$ of G such that $\langle \chi_i, \chi_{i+1} \rangle_{\mathscr{C}} \neq 0$ for all $i \in \{1, \dots, n-1\}$. The $\mathscr{C}$-blocks define a partition of $\mathrm{Irr}(G)$ (the fact that $1 \in \mathscr{C}$ ensures that each irreducible character of G belongs to a $\mathscr{C}$-block). If we take $\mathscr{C}$ to be the set of p-regular elements of G (i.e. whose order is not divisible by p), for some prime p, then the $\mathscr{C}$-blocks are just the 'ordinary' p-blocks (cf. for example [8, Theorem 3.19]).

Let $\mathrm{CF}(G)$ be the set of complex class functions of G, and $\langle\, .\,, .\, \rangle$ be the ordinary scalar product on $\mathrm{CF}(G)$. For any $\chi \in \mathrm{Irr}(G)$, we define $\chi^{\mathscr{C}} \in \mathrm{CF}(G)$ by letting

$$\chi^{\mathscr{C}}(g) = \begin{cases} \chi(g) & \text{if } g \in \mathscr{C}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $\chi \in \mathrm{Irr}(G)$, we have

$$\chi^{\mathscr{C}} = \sum_{\psi \in \mathrm{Irr}(G)} \langle \chi^{\mathscr{C}}, \psi \rangle \psi = \sum_{\psi \in \mathrm{Irr}(G)} \langle \chi, \psi \rangle_{\mathscr{C}} \psi.$$

Since $\langle \chi, \psi \rangle_{\mathscr{C}} \in \mathbb{Q}$ for all $\psi \in \mathrm{Irr}(G)$, there exists $d \in \mathbb{N}$ such that $d\chi^{\mathscr{C}}$ is a generalized character of G. We call the smallest such positive integer the *$\mathscr{C}$-defect* of χ, and denote it by $d_{\mathscr{C}}(\chi)$.

It is easy to check that $\chi \in \mathrm{Irr}(G)$ has $\mathscr{C}$-defect 1 if and only if χ vanishes outside $\mathscr{C}$. This is also equivalent to the fact that $\{\chi\}$ is a $\mathscr{C}$-block of G.

Writing 1_G for the trivial character of G, we see that, for any $\chi \in \mathrm{Irr}(G)$, $d_{\mathscr{C}}(1_G)\chi^{\mathscr{C}} = \chi \otimes (d_{\mathscr{C}}(1_G)1_G^{\mathscr{C}})$ is a generalized character, so that $d_{\mathscr{C}}(\chi)$ divides $d_{\mathscr{C}}(1_G)$. In particular, 1_G has maximal $\mathscr{C}$-defect.

Note that, if $\mathscr{C}$ is the set of p-regular elements of G (p a prime), then, for all $\chi \in \mathrm{Irr}(G)$, we have (cf. for example [8, Lemma 3.23])

$$d_{\mathscr{C}}(\chi) = \left(\frac{|G|}{\chi(1)} \right)_p = p^{d(\chi)},$$

where $d(\chi)$ is the ordinary p-defect of χ.

One key notion defined in [6] is that of a *generalized perfect isometry*. Suppose that G and H are finite groups, and $\mathscr{C}$ and $\mathscr{D}$ are closed unions of conjugacy classes of G and H respectively. Take a union b of $\mathscr{C}$-blocks of G, and a union b' of $\mathscr{D}$-blocks of G. A *generalized perfect isometry* between b and b' (with respect to $\mathscr{C}$ and $\mathscr{D}$) is a bijection with signs between b and b', which furthermore preserves contributions. That is, $I: b \mapsto b'$ is a bijection such that, for each $\chi \in b$, there is a sign $\varepsilon(\chi)$ such that

$$\langle I(\chi), I(\psi) \rangle_{\mathscr{D}} = \langle \varepsilon(\chi)\chi, \varepsilon(\psi)\psi \rangle_{\mathscr{C}} \quad \text{for all } \chi, \psi \in b.$$

In particular, one sees that a generalized perfect isometry I preserves the defect, that is, for all $\chi \in b$, we have $d_{\mathscr{C}}(\chi) = d_{\mathscr{D}}(I(\chi))$.

Note that, if $\mathscr{C}$ and $\mathscr{D}$ are the sets of p-regular elements of G and H respectively, then this notion is a bit weaker than that of *perfect isometry* introduced by M. Broué (cf. [1]). If two p-blocks b and b' are perfectly isometric in Broué's sense, then there is a generalized perfect isometry (with respect to p-regular elements) between b and b'. It is however possible to exhibit generalized perfect isometries in some cases where there is no perfect isometry in Broué's sense (cf. [3]).

Külshammer, Olsson and Robinson defined and studied in [6] the ℓ-blocks of the symmetric group, where $\ell \geqslant 2$ is any integer. They did this by taking $\mathscr{C}$ to be the set of *ℓ-regular* elements, that is, which have no cycle (in their canonical cycle decomposition) of length divisible by ℓ (in particular, if ℓ is a prime p, then the ℓ-blocks are just the p-blocks).

In Section 2, we find the *ℓ-defect* of the characters of the symmetric group $\mathfrak{S}_n$. It turns out (Theorem 2.6) that it is given by an analogue of the hook-length formula (for the degree of a character). In Section 3, we then use this to prove, when $n < \ell^2$, an ℓ-analogue of the McKay conjecture in $\mathfrak{S}_n$ (Theorem 3.4).

2 Hook-length formula

2.1 ℓ-blocks of the symmetric group. Take two integers $1 \leqslant \ell \leqslant n$, and consider the symmetric group $\mathfrak{S}_n$ on n letters. The conjugacy classes and irreducible complex characters of $\mathfrak{S}_n$ are parametrized by the set $\{\lambda \vdash n\}$ of partitions of n. We write $\mathrm{Irr}(\mathfrak{S}_n) = \{\chi_\lambda, \lambda \vdash n\}$. An element of $\mathfrak{S}_n$ is said to be *ℓ-regular* if none of its cycles has length divisible by ℓ. We let $\mathscr{C}$ be the set of ℓ-regular elements of $\mathfrak{S}_n$. The $\mathscr{C}$-blocks of $\mathfrak{S}_n$ are called *ℓ-blocks*, and they satisfy the following:

Theorem 2.1 (Generalized Nakayama conjecture [6, Theorem 5.13]). *Two characters $\chi_\lambda, \chi_\mu \in \mathrm{Irr}(\mathfrak{S}_n)$ belong to the same ℓ-block if and only if λ and μ have the same ℓ-core.*

The proof of this goes as follows. If $\langle \chi_\lambda, \chi_\mu \rangle \neq 0$, then an induction argument using the Murnaghan–Nakayama rule shows that λ and μ must have the same ℓ-core. In particular, the partitions labeling the characters in an ℓ-block all have the same ℓ-weight, and we can talk about the *ℓ-weight* of an ℓ-block.

Conversely, let B be the set of irreducible characters of $\mathfrak{S}_n$ labeled by those partitions of n which have a given ℓ-core, γ say, and ℓ-weight w. It is a well-known combinatorial fact (cf. for example [5, Theorem 2.7.30]) that the characters in B are parametrized by the *ℓ-quotients*, which can be regarded as the set of *ℓ-tuples of partitions of w*. For $\chi_\lambda \in B$, the quotient β_λ is a sequence $(\lambda^{(1)}, \dots, \lambda^{(\ell)})$ such that, for each $1 \leqslant i \leqslant \ell$, $\lambda^{(i)}$ is a partition of some k_i, $0 \leqslant k_i \leqslant w$, and $\sum_{i=1}^{\ell} k_i = w$ (the quotient β_λ 'stores' the information about how to remove w ℓ-hooks from λ to get γ). We write $\beta_\lambda \Vdash w$. To prove that B is an ℓ-block of $\mathfrak{S}_n$, Külshammer, Olsson and Robinson use a generalized perfect isometry between B and the wreath product $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ (where $\mathbb{Z}_\ell$ denotes a cyclic group of order ℓ).

The conjugacy classes of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ are parametrized by the ℓ-tuples of partitions of w as follows (cf. [5, Theorem 4.2.8]). Write $\mathbb{Z}_\ell = \{g_1, \dots, g_\ell\}$ for the cyclic group of order ℓ. The elements of the wreath product $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ are of the form $(h; \sigma) = (h_1, \dots, h_w; \sigma)$, with $h_1, \dots, h_w \in \mathbb{Z}_\ell$ and $\sigma \in \mathfrak{S}_w$. For any such element, and for any k-cycle $\kappa = (j, j\kappa, \dots, j\kappa^{k-1})$ in σ, we define the *cycle product* of $(h; \sigma)$ and κ by

$$g((h; \sigma), \kappa) = h_j h_{j\kappa^{-1}} h_{j\kappa^{-2}} \dots h_{j\kappa^{-(k-1)}} \in \mathbb{Z}_\ell.$$

If σ has cycle structure π say, then we form ℓ partitions $(\pi_1, \dots, \pi_\ell)$ from π as follows: any cycle κ in π gives a cycle of the same length in π_i if $g((h; \sigma), \kappa) = g_i$. The resulting ℓ-tuple of partitions of w describes the *cycle structure* of $(h; \sigma)$, and two elements of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ are conjugate if and only if they have the same cycle structure. An element of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ is said to be *regular* if it has no cycle product equal to 1.

The irreducible characters of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ are also canonically parametrized by the ℓ-tuples of partitions of w in the following way. Write $\mathrm{Irr}(\mathbb{Z}_\ell) = \{\alpha_1, \dots, \alpha_\ell\}$, and take $\beta_\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \Vdash w$, with $\lambda^{(i)} \vdash k_i$ as above $(1 \leqslant i \leqslant \ell)$. The irreducible character $\alpha_1^{k_1} \otimes \cdots \otimes \alpha_\ell^{k_\ell}$ of the *base group* $\mathbb{Z}_\ell^w$ can be extended in a natural way to its inertia subgroup $(\mathbb{Z}_\ell \wr \mathfrak{S}_{k_1}) \times \cdots \times (\mathbb{Z}_\ell \wr \mathfrak{S}_{k_\ell})$, giving the irreducible character $\prod_{i=1}^{\ell} \widehat{\alpha_i^{k_i}}$. The tensor product $\prod_{i=1}^{\ell} \widehat{\alpha_i^{k_i}} \otimes \chi_{\lambda^{(i)}}$ is an irreducible character of

$$(\mathbb{Z}_\ell \wr \mathfrak{S}_{k_1}) \times \cdots \times (\mathbb{Z}_\ell \wr \mathfrak{S}_{k_\ell})$$

which extends $\prod_{i=1}^{\ell} \widehat{\alpha_i^{k_i}}$, and it remains irreducible when induced to $\mathbb{Z}_\ell \wr \mathfrak{S}_w$. We denote by χ_{β_λ} this induced character. Furthermore, any irreducible character of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ can be obtained in this way.

In [6], the authors show that the map $\chi_\lambda \mapsto \chi_{\beta_\lambda}$ is a generalized perfect isometry between B and $\mathrm{Irr}(\mathbb{Z}_\ell \wr \mathfrak{S}_w)$, with respect to ℓ-regular elements of $\mathfrak{S}_n$ and regular elements of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$.

On the other hand, they show that, writing 'reg' for the set of regular elements of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$, we have, for all $\chi \in \mathrm{Irr}(\mathbb{Z}_\ell \wr \mathfrak{S}_w)$,

$$\mathbb{Z} \ni \frac{\ell^w w! \langle \chi, 1_{\mathbb{Z}_\ell \wr \mathfrak{S}_w} \rangle_{\mathrm{reg}}}{\chi(1)} \equiv (-1)^w \pmod{\ell}, \tag{1}$$

where $1_{\mathbb{Z}_\ell \wr \mathfrak{S}_w}$ is the trivial character of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$. In particular, $\langle \chi, 1_{\mathbb{Z}_\ell \wr \mathfrak{S}_w} \rangle_{\mathrm{reg}} \neq 0$. Using the generalized perfect isometry that we described above, we see that there exists a character $\chi_\lambda \in B$ such that, for all $\chi_\mu \in B$, we have $\langle \chi_\lambda, \chi_\mu \rangle_{\mathscr{C}} \neq 0$, where $\mathscr{C}$ is the set of ℓ-regular elements of $\mathfrak{S}_n$. In particular, all characters in B belong to the same ℓ-block of $\mathfrak{S}_n$, which ends the proof of Theorem 2.1.

2.2 ℓ-defect of characters. Using the ingredients in the proof of Theorem 2.1, we can now compute explicitly the *ℓ-defects* of the irreducible characters of $\mathfrak{S}_n$ (that is, their $\mathscr{C}$-defect, where $\mathscr{C}$ is the set of ℓ-regular elements of $\mathfrak{S}_n$).

As we remarked earlier, if λ is a partition of n of ℓ-weight w, then, because of the generalized perfect isometry we described above, the ℓ-defect $d_\ell(\chi_\lambda)$ of $\chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ is the same as the reg-defect $d_{\mathrm{reg}}(\chi_{\beta_\lambda})$ of $\chi_{\beta_\lambda} \in \mathrm{Irr}(\mathbb{Z}_\ell \wr \mathfrak{S}_w)$, where β_λ is the ℓ-quotient of λ. It is in fact these reg-defects that we will compute.

First note that, if $w = 0$, then λ is its own ℓ-core, so that χ_λ is alone in its ℓ-block, and $d_\ell(\chi_\lambda) = 1$. We therefore now fix $w \geqslant 1$.

We write π the set of primes dividing ℓ. Every positive integer m can be factorized uniquely as $m = m_\pi m_{\pi'}$, where every prime factor of m_π belongs to π and no prime factor of $m_{\pi'}$ is contained in π. We call m_π the π-part of m.

Using results of Donkin (cf. [2]) and equality (1), Külshammer, Olsson and Robinson proved the following:

Theorem 2.2 ([6, Theorem 6.2]). *The* reg*-defect of the trivial character of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ is $\ell^w w!_\pi$.*

In particular, since $1_{\mathbb{Z}_\ell \wr \mathfrak{S}_w}$ has maximal reg-defect, we see that, for any $\chi \in \mathrm{Irr}(\mathbb{Z}_\ell \wr \mathfrak{S}_w)$, $d_{\mathrm{reg}}(\chi)$ is a π-number.

We can now compute the reg-defect of any irreducible character χ of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$. It turns out that it is sufficient to know the reg-contribution of χ with the trivial character, and this is given by (1). We have the following:

Proposition 2.3. *Take any integers $\ell \geqslant 2$ and $w \geqslant 1$. Then*

$$d_{\mathrm{reg}}(\chi) = \frac{\ell^w (w!)_\pi}{\chi(1)_\pi}$$

for any $\chi \in \mathrm{Irr}(\mathbb{Z}_\ell \wr \mathfrak{S}_w)$.

Proof. Take $\chi \in \mathrm{Irr}(\mathbb{Z}_\ell \wr \mathfrak{S}_w)$. Recall that, by (1),

$$\mathbb{Z} \ni \frac{\ell^w w!}{\chi(1)} \langle \chi, 1 \rangle_{\mathrm{reg}} \equiv (-1)^w \pmod{\ell}.$$

Now $d_{\mathrm{reg}}(\chi)$ is a π-number, so that $\langle \chi, 1 \rangle_{\mathrm{reg}}$ is a rational whose (reduced) denominator is a π-number. This implies that

$$\frac{\ell^w (w!)_\pi}{\chi(1)_\pi} \langle \chi, 1 \rangle_{\mathrm{reg}} \in \mathbb{Z}.$$

Furthermore, from (1), we also deduce that, for each $p \in \pi$,

$$\frac{\ell^w w!}{\chi(1)} \langle \chi, 1 \rangle_{\mathrm{reg}} \not\equiv 0 \pmod{p}.$$

Thus, for any $p \in \pi$,

$$\frac{\ell^w (w!)_\pi}{\chi(1)_\pi} \langle \chi, 1 \rangle_{\mathrm{reg}} \not\equiv 0 \pmod{p}.$$

Hence $\ell^w (w!)_\pi / \chi(1)_\pi$ is the smallest positive integer d such that $d\langle \chi, 1 \rangle_{\mathrm{reg}} \in \mathbb{Z}$. This implies that $\ell^w (w!)_\pi / \chi(1)_\pi$ divides $d_{\mathrm{reg}}(\chi)$ (indeed, by definition, $d_{\mathrm{reg}}(\chi)\langle \chi, 1 \rangle_{\mathrm{reg}} \in \mathbb{Z}$, and $d_{\mathrm{reg}}(\chi)$ is a π-number).

Now, conversely, if $\psi \in \mathrm{Irr}(\mathbb{Z}_\ell \wr \mathfrak{S}_w)$, then $\langle \chi, \psi \rangle_{\mathrm{reg}} \in \mathbb{Q}$, so (since $\chi(1)$ divides $|\mathbb{Z}_\ell \wr \mathfrak{S}_w| = \ell^w w!$) we also have

$$\frac{\ell^w w!}{\chi(1)} \langle \chi, \psi \rangle_{\mathrm{reg}} \in \mathbb{Q}.$$

However,

$$\frac{\ell^w w!}{\chi(1)} \langle \chi, \psi \rangle_{\mathrm{reg}} = \frac{\ell^w w!}{\ell^w w!} \sum_{g \in \mathrm{reg}/\sim} \frac{K_g \chi(g)}{\chi(1)} \psi(g^{-1})$$

(where the sum is taken over representatives for the regular classes, and, for g such a representative, K_g is the size of the conjugacy class of g). Moreover, for each g in the sum, $K_g\chi(g)/\chi(1)$ and $\psi(g^{-1})$ are both algebraic integers. Hence $(\ell^w w!/\chi(1))\langle\chi,\psi\rangle_{\mathrm{reg}}$ is also an algebraic integer, and thus an integer. Hence

$$\frac{\ell^w w!}{\chi(1)}\langle\chi,\psi\rangle_{\mathrm{reg}} \in \mathbb{Z} \quad \text{for all } \psi \in \mathrm{Irr}(\mathbb{Z}_\ell \wr \mathfrak{S}_w).$$

and this implies that $d_{\mathrm{reg}}(\chi)$ divides $\ell^w w!/\chi(1)$, and, $d_{\mathrm{reg}}(\chi)$ being a π-number, $d_{\mathrm{reg}}(\chi)$ divides $\ell^w(w!)_\pi/\chi(1)_\pi$. Hence we finally get $d_{\mathrm{reg}}(\chi) = \ell^w(w!)_\pi/\chi(1)_\pi$. $\square$

We want to express the ℓ-defect of a character in terms of hook lengths. For any $\lambda \vdash n$, we write $\mathscr{H}(\lambda)$ for the set of hooks in λ, and $\mathscr{H}_\ell(\lambda)$ for the set of hooks in λ whose length is divisible by ℓ. Similarly, if $\beta_\lambda = (\lambda^{(1)},\ldots,\lambda^{(\ell)}) \Vdash w$, we define a *hook* in β_λ to be a hook in any of the $\lambda^{(i)}$'s, and write $\mathscr{H}(\beta_\lambda)$ for the set of hooks in β_λ. Finally, for any hook h (in a partition or a tuple of partitions), we write $|h|$ for the length of h.

We will use the following classical results about hooks (cf. for example [5, §2.3, §2.7]).

Theorem 2.4. *Let $n \geqslant \ell \geqslant 2$ be any two integers, and let λ be any partition of n. Then the following assertions hold*:

(i) (Hook-length formula, [5, Theorem 2.3.21]) *We have*

$$\frac{|\mathfrak{S}_n|}{\chi_\lambda(1)} = \prod_{h\in\mathscr{H}(\lambda)} |h|.$$

(ii) ([5, 2.7.40]) *If λ has ℓ-weight w, then $|\mathscr{H}_\ell(\lambda)| = w$.*

(iii) ([5, Lemma 2.7.13 and Theorem 2.7.16]) *If β_λ is the ℓ-quotient of λ, then $\{|h|, h\in\mathscr{H}_\ell(\lambda)\} = \{\ell|h'|, h'\in\mathscr{H}(\beta_\lambda)\}$.*

We can now establish the following:

Proposition 2.5. *If $n \geqslant \ell \geqslant 2$ are integers, π is the set of primes dividing ℓ, and $\lambda \vdash n$ has ℓ-weight $w \neq 0$ and ℓ-quotient β_λ, then*

$$\frac{\ell^w(w!)_\pi}{\chi_{\beta_\lambda}(1)_\pi} = \prod_{h\in\mathscr{H}_\ell(\lambda)} |h|_\pi.$$

Proof. Write $\beta_\lambda = (\lambda^{(1)},\ldots,\lambda^{(\ell)})$, where $\lambda^{(i)} \vdash k_i$ for $1 \leqslant i \leqslant \ell$. First note that, by construction of χ_{β_λ}, and since the irreducible characters of $\mathbb{Z}_\ell$ all have degree 1, we have

$$\chi_{\beta_\lambda}(1) = \frac{\ell^w w!}{\prod_{i=1}^{\ell}\ell^{k_i}k_i!}\chi_{\lambda^{(1)}}(1)\ldots\chi_{\lambda^{(\ell)}}(1).$$

Thus, by the hook-length formula (Theorem 2.4 (i)),

$$\chi_{\beta_\lambda}(1) = \frac{w!}{\prod_{h \in \mathscr{H}(\beta_\lambda)} |h|} \quad \text{and} \quad \frac{|\mathbb{Z}_\ell \wr \mathfrak{S}_w|}{\chi_{\beta_\lambda}(1)} = \ell^w \prod_{h \in \mathscr{H}(\beta_\lambda)} |h|.$$

We therefore get

$$\frac{\ell^w (w!)_\pi}{\chi_{\beta_\lambda}(1)_\pi} = \frac{|\mathbb{Z}_\ell \wr \mathfrak{S}_w|_\pi}{\chi_{\beta_\lambda}(1)_\pi} = \ell^w \prod_{h \in \mathscr{H}(\beta_\lambda)} |h|_\pi.$$

Now, by Theorem 2.4 (ii) and (iii), we have $|\mathscr{H}(\beta_\lambda)| = w$, so that

$$\ell^w \prod_{h \in \mathscr{H}(\beta_\lambda)} |h| = \prod_{h \in \mathscr{H}(\beta_\lambda)} \ell |h|,$$

and, by Theorem 2.4 (iii), $\prod_{h \in \mathscr{H}(\beta_\lambda)} \ell |h| = \prod_{h \in \mathscr{H}_\ell(\lambda)} |h|$. Taking π-parts, we obtain $\ell^w (w!)_\pi / \chi_{\beta_\lambda}(1)_\pi = \prod_{h \in \mathscr{H}_\ell(\lambda)} |h|_\pi$, as announced. $\square$

Combining Propositions 2.3 and 2.5, we finally get

Theorem 2.6. *Let $n \geqslant \ell \geqslant 2$ be integers, and let B be an ℓ-block of $\mathfrak{S}_n$ of weight w. Then the following assertions hold.*

(i) *If $w = 0$, then $B = \{\chi_\lambda\}$ for some partition λ of n, and $d_\ell(\chi_\lambda) = 1$.*
(ii) *If $w > 0$, and if $\chi_\lambda \in B$, then $d_\ell(\chi_\lambda) = \prod_{h \in \mathscr{H}_\ell(\lambda)} |h|_\pi$, where π is the set of primes dividing ℓ (that is, $d_\ell(\chi_\lambda)$ is the π-part of the product of the hook lengths divisible by ℓ in λ).*

3 McKay conjecture

3.1 McKay conjecture, generalization. In this section, we want to study an ℓ-analogue of the following:

Conjecture 3.1 (McKay). Let G be a finite group, p be a prime, and P be a Sylow p-subgroup of G. Then the numbers of irreducible complex characters whose degree is not divisible by p are the same for G and $N_G(P)$.

The McKay conjecture was proved by Olsson [9] for the symmetric group. In order to generalize this to an arbitrary integer ℓ, we will use the results of [4], which we summarize here. Let $n \geqslant \ell \geqslant 2$ be integers. Suppose furthermore that $n < \ell^2$, and write $n = \ell w + r$, with $0 \leqslant w, r < \ell$. We define a *Sylow ℓ-subgroup* of $\mathfrak{S}_n$ to be any subgroup of $\mathfrak{S}_n$ generated by w disjoint ℓ-cycles. In particular, if ℓ is a prime p, then the Sylow ℓ-subgroups of $\mathfrak{S}_n$ are just its Sylow p-subgroups. Then any two Sylow ℓ-subgroups of $\mathfrak{S}_n$ are conjugate, and they are Abelian. Let $\mathscr{L}$ be a Sylow ℓ-subgroup of $\mathfrak{S}_n$. In

[4], the notion of an *ℓ-regular* element is given, which coincides with the notion of a p-regular element if ℓ is a prime p. Using this, one can construct the *ℓ-blocks* of $N_{\mathfrak{S}_n}(\mathscr{L})$, and show that they satisfy an analogue of Broué's Abelian defect conjecture (cf. [4, Theorem 4.1]). We will show that, still in the case where $n < \ell^2$, an analogue of the McKay conjecture also holds. However, if we just replace p by an arbitrary integer ℓ, and consider irreducible characters of degree not divisible by ℓ, or even coprime to ℓ, then the numbers differ in $\mathfrak{S}_n$ and $N_{\mathfrak{S}_n}(\mathscr{L})$. Instead, we will use the notion of ℓ-defect, and prove that the numbers of irreducible characters of maximal ℓ-defect are the same in $\mathfrak{S}_n$ and $N_{\mathfrak{S}_n}(\mathscr{L})$ (note that, if ℓ is a prime, then both statements coincide).

3.2 Defect and weight. In order to study characters of $\mathfrak{S}_n$ of maximal ℓ-defect, we need the following result, which tells us where to look for them:

Proposition 3.2. *Let $\ell \geqslant 2$ and $0 \leqslant w, r < \ell$ be integers, and let λ be a partition of $n = \ell w + r$. If $\chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ has maximal ℓ-defect, then λ has (maximal) ℓ-weight w.*

Proof. First note that, if ℓ is a prime, then this can be proved in a purely arithmetic way (cf. [7]). This does not seem to be the case when ℓ is no longer a prime, and we will use the abacus instead. For a complete description of the abacus, we refer to [5, §2.7] (note however that the abacus we use here is the horizontal mirror image of that described by James and Kerber).

Suppose, for a contradiction, that λ has ℓ-weight $v < w$. By the previous section, the ℓ-defect of χ_λ is the π-part of the product of the hook lengths divisible by ℓ in λ. Now these are visible on the ℓ-abacus of λ. This has ℓ runners, and a hook of length $k\ell$ ($k \geqslant 1$) corresponds to a bead situated on a runner, k places above an empty spot. In particular, the (ℓ)-hooks (i.e. those whose length is divisible by ℓ) in λ are stored on at most v runners. To establish the result, we will construct a partition μ of n of weight w, and such that $d_\ell(\chi_\mu) > d_\ell(\chi_\lambda)$.

Start with the ℓ-abacus of any partition ν of r. On the (at most) v runners used by λ, take some beads up to encode the same (ℓ)-hooks as for λ. Then, on $w - v$ of the (at least) $\ell - v > w - v$ remaining runners, take the highest bead one place up. The resulting abacus then corresponds to a partition of $n = r + \ell w = r + \ell v + \ell(w - v)$, and we see that $d_\ell(\chi_\mu) = \ell^{w-v} d_\ell(\chi_\lambda)$ (indeed, the (ℓ)-hooks in μ are precisely those in λ, together with $w - v$ hooks of length ℓ). This proves the result. $\square$

3.3 Generalized perfect isometry. We describe here the analogue of Broué's Abelian defect conjecture given in [4, Theorem 4.1]. We take any integers $\ell \geqslant 2$ and $0 \leqslant w, r < \ell$, and $G = \mathfrak{S}_{\ell w + r}$. We take an Abelian *Sylow ℓ-subgroup* $\mathscr{L}$ of G; that is, $\mathscr{L} \cong \mathbb{Z}_\ell^w$ is generated by w disjoint ℓ-cycles. Then $\mathscr{L}$ is a natural subgroup of $\mathfrak{S}_{\ell w}$, and we have $N_G(\mathscr{L}) \cong N_{\mathfrak{S}_{\ell w}}(\mathscr{L}) \times \mathfrak{S}_r$ and $\mathrm{Irr}(N_G(\mathscr{L})) = \mathrm{Irr}(N_{\mathfrak{S}_{\ell w}}(\mathscr{L})) \otimes \mathrm{Irr}(\mathfrak{S}_r)$. Now $N_{\mathfrak{S}_{\ell w}}(\mathscr{L}) \cong N \wr \mathfrak{S}_w = N_{\mathfrak{S}_\ell}(L) \wr \mathfrak{S}_w$, where $L = \langle \pi \rangle \cong \mathbb{Z}_\ell$ is (a subgroup of $\mathfrak{S}_\ell$) generated by a single ℓ-cycle. As in the sketch of the proof of Theorem 2.1, we see that the conjugacy classes and irreducible characters of $N_{\mathfrak{S}_{\ell w}}(\mathscr{L})$ are parametrized by the s-tuples of partitions of w, where s is the number of conjugacy classes of N.

Among these, there is a unique conjugacy class of ℓ-cycles, for which we take representative π. We take representatives $\{g_1 = \pi, g_2, \dots, g_s\}$ for the conjugacy classes of N. Considering as *ℓ-regular* any element of N not conjugate to the ℓ-cycle π, we can construct the *ℓ-blocks* of N, and show that the principal ℓ-block contains ℓ characters, which we label $\psi_1, \dots, \psi_\ell$, and that each of the remaining $s - \ell$ characters, labeled $\psi_{l+1}, \dots, \psi_s$, is alone in its ℓ-block (cf. [4, §2]). Using the construction presented after Theorem 2.1, we label the conjugacy classes and irreducible characters of $N \wr \mathfrak{S}_w$ by the s-tuples of partitions of w. An element of $N \wr \mathfrak{S}_w$ of cycle type $(\pi_1, \dots, \pi_s) \Vdash w$ is called *ℓ-regular* if $\pi_1 = \emptyset$ (and *ℓ-singular* otherwise). Then one shows that the ℓ-blocks of $N \wr \mathfrak{S}_w$ are the principal ℓ-block, $b_0 = \{\chi^\alpha, \alpha = (\alpha_1, \dots, \alpha_\ell, \emptyset, \dots, \emptyset) \Vdash w\}$, and blocks of size 1, $\{\chi^\alpha\}$, whenever $\alpha \Vdash w$ is such that $\alpha_k \neq \emptyset$ for some $\ell < k \leqslant s$ (see [4, Theorem 3.7 and Corollary 3.11]).

Finally, an element of $N_G(\mathscr{L}) \cong N_{\mathfrak{S}_{\ell w}}(\mathscr{L}) \times \mathfrak{S}_r$ is said to be *ℓ-regular* if its $N_{\mathfrak{S}_{\ell w}}(\mathscr{L})$-part is ℓ-regular in the above sense (so that, if ℓ is a prime p, then the notions of ℓ-regular and p-regular coincide). We can summarize the results of [4] as follows:

Theorem 3.3 ([4, Theorem 4.1]). *Let the notation be as above. Then any ℓ-block of $N_G(\mathscr{L})$ has size* 1 *or belongs to* $\{b_0 \otimes \{\psi\}, \psi \in \mathrm{Irr}(\mathfrak{S}_r)\}$. *Furthermore, for any $\psi \in \mathrm{Irr}(\mathfrak{S}_r)$, there is a generalized perfect isometry (with respect to ℓ-regular elements) between $b_0 \otimes \{\psi\}$ and B_ψ, where B_ψ is the ℓ-block of $\mathfrak{S}_{\ell w+r}$ consisting of the irreducible characters labeled by partitions with ℓ-core ψ.*

Note that any partition of r does appear as ℓ-core of a partition of $\ell w + r$ (for example, if $\gamma \vdash r$, then γ is the ℓ-core of $(\gamma, 1^{\ell w}) \vdash \ell w + r$).

3.4 Analogues of the McKay conjecture. We can now give the analogue of the McKay conjecture that we announced. As before, let $\ell \geqslant 2$ and $0 \leqslant w, r < \ell$ be integers, $n = \ell w + r$, and $\mathscr{L}$ be an Abelian Sylow ℓ-subgroup of $\mathfrak{S}_n$. By Proposition 3.2, any irreducible character of $\mathfrak{S}_n$ of maximal ℓ-defect has (maximal) ℓ-weight w, hence belongs to some block B_ψ, with $\psi \in \mathrm{Irr}(\mathfrak{S}_r)$. Since any generalized perfect isometry preserves the defect, Theorem 3.3 provides a bijection between the sets of irreducible characters of maximal ℓ-defect and ℓ-weight w of $\mathfrak{S}_n$ and of characters of maximal ℓ-defect in $N_{\mathfrak{S}_n}(\mathscr{L})$. We therefore obtain the following result:

Theorem 3.4. *With the above notation, the numbers of irreducible characters of maximal ℓ-defect are the same in $\mathfrak{S}_n$ and $N_{\mathfrak{S}_n}(\mathscr{L})$.*

Remark. Furthermore, we have an explicit bijection, essentially given by taking ℓ-quotients of partitions.

In fact, Theorem 3.3 gives something a bit stronger, namely:

Theorem 3.5. *For any ℓ-defect $\delta \neq 1$, there is a bijection between the set of irreducible characters of $\mathfrak{S}_n$ of ℓ-weight w and ℓ-defect δ and the set of irreducible characters of $N_{\mathfrak{S}_n}(\mathscr{L})$ of ℓ-defect δ.*

Now, McKay's conjecture is stated (and, in the case of symmetric groups, proved) without any hypothesis on the Sylow p-subgroups. One would therefore want to generalize the above results to the case where $n \geqslant \ell^2$. Examples seem to indicate that such analogues do indeed hold in this case, and that a bijection is given by taking, not only the ℓ-quotient of a partition, but its ℓ-*tower* (cf. [9]).

In order to prove these results, one would first need to generalize Proposition 3.2, showing that, for any $n \geqslant \ell \geqslant 2$, if $\chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ has maximal ℓ-defect, then λ has maximal ℓ-weight, but also maximal ℓ^2-weight, maximal ℓ^3-weight, and so on. If ℓ is a prime, then this is known to be true (cf. [7]). However, it seems hard to prove in general, even when $n = \ell^2$. The particular case where ℓ is square-free is much easier.

Also, one would need to generalize the results of [4], while making sure that, when ℓ is a prime p, the notions of ℓ-regular and p-regular elements still coincide.

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